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ON THE IMBEDDING OF A THIN RIGID BODY IN A PLASTIC MEDIUM WITH HARDENING

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The problem of the imbedding of a solid, thin, well-lubricated cutting edge in a half-space of rigidly plastic hardening material under plane strain conditions is considered in a linear formulation. It is assumed that translational hardening [1] occurs. The problem turns out to be kinematically determinate.

Directing the coordinate axes as shown in Fig. 1, a, let us write the equation of the cutting edge surface as

$$y = \delta f(x) \qquad (1)$$

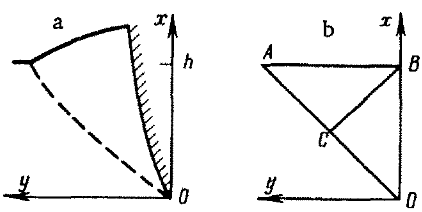

Fig. 1.

where δ is a small dimensionless parameter, and f is a sufficiently smooth function. At the initial instant the material occupies the half-space $x \leqslant 0$. Reversing the motion, let us consider the cutting edge fixed, and the medium to be displaced progressively upward along the x axis with some constant velocity u°. As a result of imbedding the cutting edge, the plastic material buckles [warps] to form a certain surface whose equation is

$$x - h = \delta\varphi(y) \tag{2}$$

where h is the depth of the imbedding (Fig. 1, a).

To solve the problem, let us use the equilibrium equation, the plasticity condition, and the relationships of the associated plastic flow law

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau}{\partial y} = 0, \qquad \frac{\partial\tau}{\partial x} + \frac{\partial\sigma_y}{\partial y} = 0 \qquad (\tau = \tau_{xy}) \tag{3}$$

$$[(\sigma_x - ce_x) - (\sigma_y - ce_y)]^2 + 4(\tau - ce_{xy})^2 = 4k^2 \quad (c,\ k = \text{const}) \tag{4}$$

$$\varepsilon_x + \varepsilon_y = 0, \qquad \frac{2\varepsilon_x}{(\sigma_x - ce_x) - (\sigma_y - ce_y)} = \frac{\varepsilon_{xy}}{\tau - ce_{xy}} \tag{5}$$

Here e_x, e_y, e_{xy} are the strain components, ε_x, ε_y, ε_{xy} are the strain rate components. The boundary conditions will be examined below.

Considering the strains small, we have

$$\varepsilon_x = \frac{\partial e_x}{\partial t}, \qquad \varepsilon_y = \frac{\partial e_y}{\partial t}, \qquad \varepsilon_{xy} = \frac{\partial e_{xy}}{\partial t} \tag{6}$$

$$e_x = \frac{\partial s_x}{\partial x}, \qquad e_y = \frac{\partial s_y}{\partial y}, \qquad e_{xy} = \frac{1}{2}\left(\frac{\partial s_x}{\partial y} + \frac{\partial s_y}{\partial x}\right) \tag{7}$$

$$u = \frac{\partial s_x}{\partial t}, \qquad v = \frac{\partial s_y}{\partial t} \tag{8}$$

where s_x, s_y are the components of the displacement vector of points of the medium, and u, v are components of the displacement velocity.

Let us examine linearization in the small parameter δ

$$\sigma_x = \sigma_x^\circ + \delta\sigma_x', \ldots, v = v^\circ + \delta v' \tag{9}$$

The zero approximation corresponds to the unperturbed state, i.e., to imbedding of a cutting edge of zero thickness ($\delta = 0$). The surface of the medium is hence not deformed ($x = h$). The first approximation (9) becomes

$$\begin{aligned} &\sigma_x = \delta\sigma_x', \qquad \sigma_y = -2k + \delta\sigma_y', \qquad \tau = \delta\tau' \\ &e_x = \delta e_x', \qquad e_y = \delta e'_y, \qquad e_{xy} = \delta e'_{xy} \\ &u = u^\circ + \delta u', \qquad v = \delta v' \qquad (u^\circ = \text{const} \neq 0) \end{aligned} \tag{10}$$

Linearizing (4), (5), we obtain

$$\sigma_x' - \sigma_y' = c(e_x' - e_y') \tag{11}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0, \qquad \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} = 0 \tag{12}$$

Let us consider the kinematic boundary conditions. The first is that the velocities on the cutting edge surface are directed along the tangent to the cutting edge, therefore

$$\mathbf{v}\cdot\mathbf{n} = 0 \tag{13}$$

$$\mathbf{v} = (u^\circ + \delta u')\,\mathbf{i} + \delta v'\mathbf{j}, \qquad \mathbf{n} = \delta\frac{\partial f(x)}{\partial x}\,\mathbf{i} - \mathbf{j}$$

where $\mathbf{n}$ is the unit vector normal to the cutting edge surface. To higher order accuracy, we obtain from (13)

$$v' = u^\circ \frac{\partial f(x)}{\partial x} \quad \text{for} \quad y = 0 \tag{14}$$

The boundary of the plastic material in the zeroth approximation is the line $x - y = 0$, and in the first approximation is the line

$$x - y + \delta\gamma = 0 \tag{15}$$

where γ is some sufficiently smooth function. The unit vector normal to the boundary is to first order accuracy

$$\mathbf{v} = \frac{1}{\sqrt{2}} (\mathbf{i} - \mathbf{j}) \tag{16}$$

Since the motion has been reversed and the cutting edge is fixed, the normal velocity on the boundary of the plastic material equals $u^\circ / \sqrt{2}$ to second order accuracy. Therefore

$$\mathbf{v} \cdot \mathbf{v} = \frac{u^\circ}{\sqrt{2}} \quad \text{for} \quad x - y + \delta\gamma = 0 \tag{17}$$

From (17), (16) and the second relationship in (13) we obtain

$$u' - v' = 0 \quad \text{for} \quad x - y = 0 \tag{18}$$

The solution of (12) with the boundary conditions (14) and (18) is

$$u' = v' = u^\circ \frac{\partial f(x - y)}{\partial x} \tag{19}$$

i.e., the velocity field is such as if the medium were ideally plastic [2].

Let us consider the displacement vector of points of the medium

$$\mathbf{s} = s_x \mathbf{i} + s_y \mathbf{j}$$

It follows from (8), (10) and (19) that

$$s_x = \int_0^t u\,dt = h + \delta f(x - y), \qquad s_y = \int_0^t v\,dt = \delta f(x - y) \tag{20}$$

According to (7), (20) and (10) the strains are

$$e_x' = -e_y' = \frac{\partial f(x - y)}{\partial x}, \qquad e_{xy}' = 0 \tag{21}$$

Because of (21) the relationship (11) becomes

$$\sigma_x' - \sigma_y' = 2c \frac{\partial f(x - y)}{\partial x} \tag{22}$$

Upon imbedding the cutting edge, points on the surface of the medium acquire the displacements $s_x(h, y)$, $s_y(h, y)$, hence, points with the coordinates $s_x(h, y)$, $y + s_y(h, y)$ should lie on the buckled surface of the medium. Substituting these coordinates into (2), and linearizing, we obtain

$$\varphi(y) = f(h - y), \qquad 0 \leqslant y \leqslant h \tag{23}$$

Therefore, the buckled surface of plastic material coincides with the shape of the surface of the imbedded cutting edge.

Let us turn to the determination of the stress field. Because of (10) and (22) the equilibrium equations (3) become

$$\frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau'}{\partial y} = 0, \qquad \frac{\partial \tau'}{\partial x} + \frac{\partial \sigma_x'}{\partial y} + 2c \frac{\partial^2 f(x - y)}{\partial x^2} = 0 \tag{24}$$

Substituting

$$\sigma_x' = \frac{\partial U}{\partial y}, \qquad \tau' = -\frac{\partial U}{\partial x} \tag{25}$$

into (24), we arrive at the wave equation whose general solution is

$$U = Q_1(x+y) - Q_2(x-y) + \frac{c}{2}(x+y)\frac{\partial f(x-y)}{\partial x} \tag{26}$$

where Q_1 and Q_2 are arbitrary functions. Differentiating (26), we have in conformity with (25)

$$\sigma_x' = q_1(x+y) + q_2(x-y) + \frac{c}{2}\frac{\partial f(x-y)}{\partial x} - \frac{c}{2}(x+y)\frac{\partial^2 f(x-y)}{\partial x^2}$$

$$\tau' = -q_1(x+y) + q_2(x-y) - \frac{c}{2}\frac{\partial f(x-y)}{\partial x} - \frac{c}{2}(x+y)\frac{\partial^2 f(x-y)}{\partial x^2} \tag{27}$$

$$q_1(x+y) = \partial Q_1(x+y)/\partial x, \qquad q_2(x-y) = \partial Q_2(x-y)/\partial x$$

The functions q_1 and q_2 are determined from the boundary conditions for the stresses, which require that there are no tangential stresses on the surface of the smooth cutting edge, and the buckled surface of the material is stress-free.

According to [2], the linearized boundary conditions are

$$\tau' = 2k\frac{\partial f(x)}{\partial x} \quad \text{for} \quad y = 0, \ 0 \leqslant x \leqslant h \tag{28}$$

$$\sigma_x' = 0, \qquad \tau' = -2k\frac{\partial f(h-y)}{\partial y} \quad \text{for} \quad x = h, \ 0 \leqslant y \leqslant h$$

Using the boundary conditions (28), we have from (27)

$$-q_1(x) + q_2(x) = \frac{c}{2}\frac{\partial f(x)}{\partial x} + \frac{c}{2}x\frac{\partial^2 f(x)}{\partial x^2} + 2k\frac{\partial f(x)}{\partial x}, \quad 0 \leqslant x \leqslant h$$

$$q_1(h+y) + q_2(h-y) = -\frac{c}{2}\frac{\partial f(h-y)}{\partial x} + \frac{c}{2}(h+y)\frac{\partial^2 f(h-y)}{\partial x^2} \tag{29}$$

$$-q_1(h+y) + q_2(h-y) = \frac{c}{2}\frac{\partial f(h-y)}{\partial x} + \frac{c}{2}(h+y)\frac{\partial^2 f(h-y)}{\partial x^2} +$$

$$+ 2k\frac{\partial f(h-y)}{\partial x}, \quad 0 \leqslant y \leqslant h$$

Let us introduce the notation

$$h - y = \xi, \quad 0 \leqslant \xi \leqslant h \quad h + y = \eta, \quad h \leqslant \eta \leqslant 2h$$

Then from the last two relations in (29), we obtain

$$q_2(\xi) = \frac{c}{2}(2h-\xi)\frac{\partial^2 f(\xi)}{\partial \xi^2} + k\frac{\partial f(\xi)}{\partial \xi}, \qquad 0 \leqslant \xi \leqslant h \tag{30}$$

$$q_1(\eta) = -\left(k + \frac{c}{2}\right)\frac{\partial f(2h-\eta)}{\partial \eta}, \qquad h \leqslant \eta \leqslant 2h \tag{31}$$

From (30) and the first relationship in (29) we have

$$q_1(\xi) = c(h-\xi)\frac{\partial^2 f(\xi)}{\partial \xi^2} - \left(k + \frac{c}{2}\right)\frac{\partial f(\xi)}{\partial \xi}, \qquad 0 \leqslant \varepsilon \leqslant h \tag{32}$$

Let us write (31) and (32) governing the function q_1 as

$$q_1(\eta) = \begin{cases} c(h-\eta)\dfrac{\partial^2 f(\eta)}{\partial \eta^2} - \left(k + \dfrac{c}{2}\right)\dfrac{\partial f(\eta)}{\partial \eta}, & 0 \leqslant \eta \leqslant h \\ -\left(k + \dfrac{c}{2}\right)\dfrac{\partial f(2h-\eta)}{\partial \eta}, & h \leqslant \eta \leqslant 2h \end{cases} \tag{33}$$

In conformity with (33), the plastic zone consists of two domains in which the stresses have different analytical expressions. For the domain OBC (Fig. 1, b), we obtain from the first relationship in (33), and (22)

$$\begin{aligned}\sigma_x' &= c(h-x-y)\frac{\partial^2 f(x+y)}{\partial x^2} - \left(k+\frac{c}{2}\right)\frac{\partial f(x+y)}{\partial x} + \\ &+ \left(k+\frac{c}{2}\right)\frac{\partial f(x-y)}{\partial x} + c(h-x)\frac{\partial^2 f(x-y)}{\partial x^2} \\ \sigma_y' &= c(h-x-y)\frac{\partial^2 f(x+y)}{\partial x^2} - \left(k+\frac{c}{2}\right)\frac{\partial f(x+y)}{\partial x} + \\ &+ \left(k-\frac{3c}{2}\right)\frac{\partial f(x-y)}{\partial x} + c(h-x)\frac{\partial^2 f(x-y)}{\partial x^2} \\ \tau' &= -c(h-x-y)\frac{\partial^2 f(x+y)}{\partial x^2} + \left(k+\frac{c}{2}\right)\frac{\partial f(x+y)}{\partial x} + \\ &+ \left(k-\frac{c}{2}\right)\frac{\partial f(x-y)}{\partial x} + c(h-x)\frac{\partial^2 f(x-y)}{\partial x^2}\end{aligned} \tag{34}$$

For the domain ABC, we find from (27), (30), the second relationship in (33) and (22)

$$\begin{aligned}\sigma_x' &= -\left(k+\frac{c}{2}\right)\frac{\partial f(2h-x-y)}{\partial x} + \left(k+\frac{c}{2}\right)\frac{\partial f(x-y)}{\partial x} + c(h-x)\frac{\partial^2 f(x-y)}{\partial x^2} \\ \sigma_y' &= -\left(k+\frac{c}{2}\right)\frac{\partial f(2h-x-y)}{\partial x} + \left(k-\frac{3c}{2}\right)\frac{\partial f(x-y)}{\partial x} + c(h-x)\frac{\partial^2 f(x-y)}{\partial x^2} \\ \tau' &= \left(k+\frac{c}{2}\right)\frac{\partial f(2h-x-y)}{\partial x} + \left(k-\frac{c}{2}\right)\frac{\partial f(x-y)}{\partial x} + c(h-x)\frac{\partial^2 f(x-y)}{\partial x^2}\end{aligned} \tag{35}$$

The equation of the line BC is $x+y=h$. As is seen from (34) and (35), the stresses are continuous on this line. For an ideally plastic medium $c=0$ in (34) and (35).

The influence of inertial forces [3] can be taken into account in the problem considered.

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